Quaternions and 3d rotations

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Abstract

The quaternions were discovered by Hamilton in 1843. They were a recondite branch of pure mathematics until they began to be used for specifying the smooth interpolation of rotating bodies in spacecraft dynamics, robotics and computer animation. This article explains the mathematical connection between quaternions and rotations in three dimensions, and how great circle interpolation of unit quaternions gives smooth rotation paths.

The PC program `qrot` demonstrates these concepts. It can be obtained from `www.ncvp.co.uk`.

1 Affine transformations in computer graphics

A common application in computer graphics is the projection of a three dimensional scene onto a two dimensional display surface. The objects in the scene are constructed of points, lines, surfaces and so on, which are themselves defined by sets of three dimensional vectors $\mathbf{x} = (x_0, x_2, x_2)$. The movement of the objects in the scene consists of invertible rotations followed by translations

$$\mathbf{x} \longmapsto \mathbf{xR} + \mathbf{t}$$

where $\mathbf{R}$ is a rotation matrix in $SO(3)$ and $\mathbf{t} = (t_0, t_2, t_2)$ is a translation. This transformation can be represented as a single matrix multiplication

$$(\mathbf{x} \quad 1)\ \mathbf{A} = \quad (\mathbf{xR} + \mathbf{t} \quad 1) \qquad \text{where} \quad \mathbf{A} = \begin{pmatrix} \mathbf{R} & 0 \\ \mathbf{t} & 1 \end{pmatrix} \tag{1}$$

This representation is very convenient because nested transformations are equivalent to a sequence of matrix multiplications. If an object moves (A_{obj}) on a table which itself is moving (A_{table}) in a room, the movement of the object in the scene is represented by $A_{obj}A_{table}$. In fact, the final projection of the scene onto the screen can also be expressed as a multiplication by a matrix in $M_4(\mathbb{R})$, and all this matrix arithmetic is normally performed in hardware.

It is clear from equation 1 that translation and rotation are fundamentally different transformations. They are normally handled completely separately in computer graphics applications.

2 Interpolation

Smooth movement must often be interpolated between given key positions. These keys might be samples from position encoders, calculations at discrete times in a numerical simulation, or key frames specified by an animator.

Suppose the translations of an object at two keys are given by the vectors $\mathbf{t}_0$ and $\mathbf{t}_1$, and the object must move from the first to the second as a time-like parameter, t, is varied from 0 to 1.

A natural movement is given by the simple formula

$$\mathbf{t}(t) = (1-t)\mathbf{t}_0 + t\mathbf{t}_1 \tag{2}$$

Can this method be used to interpolate rotations from $\mathbf{R}_0$ to $\mathbf{R}_1$? Term-wise interpolation of the matrices is of no use, because

$$\mathbf{R}(t) = (1-t)\mathbf{R}_0 + t\mathbf{R}_1$$

gives a matrix $\mathbf{R}(t)$ which is generally not even in $SO(3)$.

Orientations are frequently expressed in terms of rotations about the three Cartesian axes in some specified order. The sizes of these rotations are called Euler angles in programming circles, and it is possible to linearly interpolate them. Unfortunately, this does not usually produce a smooth change in orientation, as you can see by trying the program `qrot`.

3 Euler's rotation theorem

Any orientation-preserving mapping of a sphere onto itself is equivalent to a rotation about a diameter of the sphere. If the mapping is not the identity this rotation is unique (apart from the opposite rotation about the opposite axis).

Proof:
The transformation is specified by its effect on two points of the sphere. Let the transformation T map the point q to the point r, and let p be the point such that $T : p \longrightarrow q$. Then T maps the great circle arc pq onto qr. Let Π be the plane containing p, q and r, and D be the diameter of the sphere perpendicular to Π. T is represented by a rotation about D which maps the arc $pq \in \Pi$ into $qr \in \Pi$.

4 Angle-axis representation of $SO(3)$

An element of $SO(3)$ may be described as the clockwise rotation by θ about an axis specified by a unit length vector vector $\mathbf{a}$. This angle-axis $(\theta, \mathbf{a})$ representation is not unique, even if θ is limited to the interval $[0, 2\pi)$, since $(\theta, \mathbf{a}) = (2\pi - \theta, -\mathbf{a})$, but the mapping $(\theta, \mathbf{a}) \longrightarrow SO(3)$ is well defined, as follows
In figure 1 $\mathbf{p}$ is a point not on the axis of rotation and $\mathbf{q}$ is its image after a rotation of θ. $\Pi_{\mathbf{p}}$ is the plane perpendicular to the axis of rotation which contains $\mathbf{p}$ and $\mathbf{q}$. $\mathbf{r}$ is the point where $\Pi_{\mathbf{p}}$ meets the axis of rotation. $\mathbf{s}$ is the point at which the perpendicular from $\mathbf{q}$ meets the plane containing $\mathbf{0}$, $\mathbf{p}$ and $\mathbf{r}$.
From the figure

$$\begin{aligned}
\|\mathbf{p}-\mathbf{r}\| &= \|\mathbf{q}-\mathbf{r}\| = \|\mathbf{p}\|\sin\phi \\
\mathbf{s} &= \mathbf{r} + \cos\theta(\mathbf{p}-\mathbf{r}) \\
\|\mathbf{q}-\mathbf{s}\| &= \|\mathbf{p}-\mathbf{r}\|\sin\theta = \|\mathbf{p}\|\sin\theta\sin\phi \\
\mathbf{a} &= \mathbf{r}/\|\mathbf{r}\| \\
\mathbf{r} &= (\mathbf{p}\bullet\mathbf{a})\mathbf{a}
\end{aligned}$$

Then, because $\mathbf{q}-\mathbf{s}$ is parallel to $\mathbf{r}\times\mathbf{p}$

$$\mathbf{q}-\mathbf{s} = \|\mathbf{q}-\mathbf{s}\|\frac{\mathbf{r}\times\mathbf{p}}{\|\mathbf{r}\times\mathbf{p}\|}$$

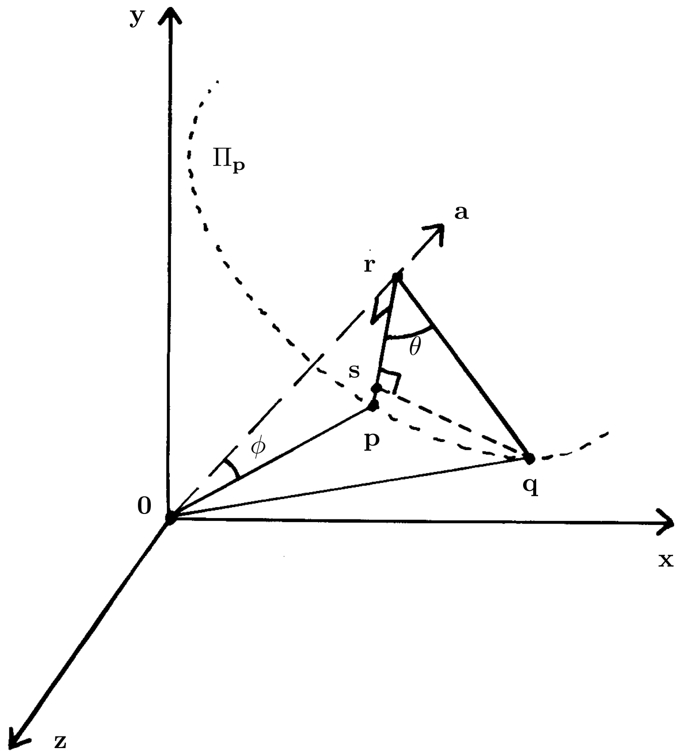

Figure 1: (from [4])

$$
\begin{aligned}
&= \|\mathbf{p}\| \sin\theta \sin\phi \frac{\mathbf{r} \times \mathbf{p}}{\|\mathbf{r}\|\|\mathbf{p}\| \sin\phi} \\
&= \sin\theta(\mathbf{a} \times \mathbf{p})
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbf{q} &= \sin\theta(\mathbf{a} \times \mathbf{p}) + \mathbf{r} + \cos\theta(\mathbf{p} - \mathbf{r}) \\
&= \sin\theta(\mathbf{a} \times \mathbf{p}) + (1 - \cos\theta)(\mathbf{p} \bullet \mathbf{a})\mathbf{a} + \cos\theta\mathbf{p} \qquad (3)
\end{aligned}
$$

To express this result in matrix terms some subsidiary results are required.

The cross product of two vectors is given by

$$
\begin{aligned}
\mathbf{x} \times \mathbf{a} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_0 & x_1 & x_2 \\ a_0 & a_1 & a_2 \end{pmatrix} \\
&= \quad (x_1 a_2 - x_2 a_1, \quad x_2 a_0 - x_0 a_2, \quad x_0 a_1 - x_1 a_0) \\
&= \mathbf{x}\mathbf{A} \qquad \text{where } \mathbf{A} = \begin{pmatrix} 0 & -a_2 & a_1 \\ a_2 & 0 & -a_0 \\ -a_1 & a_0 & 0 \end{pmatrix}
\end{aligned}
$$

The projection of a vector on a unit vector, **a**, is given by

$$
\begin{aligned}
(\mathbf{x} \bullet \mathbf{a})\mathbf{a} &= (x_0, \quad x_1, \quad x_2) \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} (a_0, \quad a_1, \quad a_2) \\
&= (x_0, \quad x_1, \quad x_2) \ \mathbf{M} \qquad \text{where } \mathbf{M} = \begin{pmatrix} a_0^2 & a_0a_1 & a_0a_2 \\ a_0a_1 & a_1^2 & a_1a_2 \\ a_0a_2 & a_1a_2 & a_2^2 \end{pmatrix}
\end{aligned}
$$

Now

$$
\mathbf{A}^2 = \begin{pmatrix} -a_1^2 - a_2^2 & a_0a_1 & a_0a_2 \\ a_0a_1 & -a_0^2 - a_2^2 & a_1a_2 \\ a_0a_2 & a_1a_2 & -a_0^2 - a_1^2 \end{pmatrix}
$$

hence

$$
(\mathbf{x} \bullet \mathbf{a})\mathbf{a} = \mathbf{x}(\mathbf{I} + \mathbf{A}^2)
$$

So equation 3 becomes

$$
\begin{aligned}
\mathbf{q} &= \mathbf{pA}\sin\theta + \mathbf{p}(\mathbf{I} + \mathbf{A}^2)(1 - \cos\theta) + \mathbf{pI}\cos\theta \\
&= \mathbf{pA}\sin\theta + \mathbf{pI} + \mathbf{pA}^2(1 - \cos\theta)
\end{aligned}
$$

Hence $\mathbf{q} = \mathbf{pR_a}(\theta)$, where

$$
\mathbf{R_a}(\theta) = \mathbf{I} + \mathbf{A}\sin\theta + \mathbf{A}^2(1 - \cos\theta)
$$

This is a convenient form for calculations.

5 $\mathbf{R_a}(\theta)$ is a 1-parameter subgroup of $SO(3)$

A further property of $\mathbf{A}$ is

$$
\mathbf{A}^3 = \begin{pmatrix} a_0a_1a_2 - a_0a_1a_2 & a_2(a_0^2 + a_2^2) + a_2a_1^2 & -a_1(a_0^2 + a_1^2) - a_1a_2^2 \\ -a_2(a_1^2 + a_2^2) - a_2a_0^2 & a_0a_1a_2 - a_0a_1a_2 & a_0(a_0^2 + a_1^2) + a_0a_2^2 \\ a_1(a_1^2 + a_2^2) + a_1a_0^2 & a_0(a_0^2 + a_2^2) + a_0a_1^2 & a_0a_1a_2 - a_0a_1a_2 \end{pmatrix} = -\mathbf{A}
$$

hence

$$
\mathbf{A}^n = \begin{cases} (-1)^{(n-1)/2}\mathbf{A} & \text{if } n \text{ is odd} \\ (-1)^{(n-2)/2}\mathbf{A}^2 & \text{if } n \text{ is even} \end{cases}
$$

Now

$$
\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \quad \ldots
$$

and

$$1 - \cos\theta = \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} \quad \ldots$$

so

$$\begin{aligned}\sin\theta\mathbf{A} &= \theta\mathbf{A} - \frac{\theta^3}{3!}\mathbf{A} + \frac{\theta^5}{5!}\mathbf{A} - \frac{\theta^7}{7!}\mathbf{A} \quad \ldots \\ &= \theta\mathbf{A} + \frac{\theta^3}{3!}\mathbf{A}^3 + \frac{\theta^5}{5!}\mathbf{A}^5 + \frac{\theta^7}{7!}\mathbf{A}^7 \quad \ldots\end{aligned}$$

and

$$\begin{aligned}1 - \cos\theta\mathbf{A} &= \frac{\theta^2}{2!}\mathbf{A} - \frac{\theta^4}{4!}\mathbf{A} + \frac{\theta^6}{6!}\mathbf{A} \quad \ldots \\ &= \frac{\theta^2}{2!}\mathbf{A}^2 + \frac{\theta^4}{4!}\mathbf{A}^4 + \frac{\theta^6}{6!}\mathbf{A}^6 \quad \ldots\end{aligned}$$

So

$$\begin{aligned}\mathbf{R_a}(\theta) &= \mathbf{I} + \mathbf{A}\sin\theta + \mathbf{A}^2(1 - \cos\theta) \\ &= \mathbf{I} + \theta\mathbf{A} + \frac{\theta^2}{2!}\mathbf{A}^2 + \frac{\theta^3}{3!}\mathbf{A}^3 + \frac{\theta^4}{4!}\mathbf{A}^4 \quad \ldots \\ &= \exp(\theta\mathbf{A})\end{aligned}$$

6 Quaternions represent rotations

A quaternion has the form $\mathbf{q} = w + xi + yj + zk$, where w, x, y and z are real numbers, $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, and $ki = j = -ik$. The set of all quaternions is called $\mathbb{H}$ after Hamilton.

The inner product of two quaternions is $\mathbf{q}_1 \bullet \mathbf{q}_2 = w_1w_2 + x_1x_2 + y_1y_2 + z_1z_2$, so the norm of a quaternion $\|\mathbf{q}\|$ is $(\mathbf{q} \bullet \mathbf{q})^{1/2}$.

The set of unit quaternions $S^3 = \{\mathbf{q} : \|\mathbf{q}\| = 1\}$ is the surface of a sphere in $\mathbb{R}^4$.

A unit quaternion may be expressed in the form $\mathbf{q}(\theta, \mathbf{a}) = \cos\theta + \sin\theta\hat{\mathbf{a}}$, where $\hat{\mathbf{a}} = a_0i + a_1j + a_2k$, and $\|\hat{\mathbf{a}}\| = 1$.
$\hat{\mathbf{a}}$ and $\mathbf{a}$ may be identified, and sometimes $\hat{\mathbf{a}}$ will appear as both a vector and as a pure quaternion. Notice that

$$\begin{aligned}\hat{\mathbf{a}}\hat{\mathbf{a}} &= (u_0i + u_1j + u_2k)(u_0i + u_1j + u_2k) \\ &= -u_0^2 - u_1^2 - u_2^2 + (u_1u_2 - u_1u_2)i + (u_0u_2 - u_0u_2)j + (u_0u_1 - u_0u_1)k = -1\end{aligned}$$

so care must be used when computing quaternion products in this form.

There is a useful identity like Euler's for complex numbers

$$
\begin{aligned}
\exp(\theta\hat{\mathbf{a}}) &= 1 + \theta\hat{\mathbf{a}} + \frac{\theta^2}{2!}\hat{\mathbf{a}}^2 + \frac{\theta^3}{3!}\hat{\mathbf{a}}^3 + \dots \\
&= \theta\hat{\mathbf{a}} - \frac{\theta^3}{3!}\hat{\mathbf{a}} + \frac{\theta^5}{5!}\hat{\mathbf{a}} + \dots \\
& \quad 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots \\
&= \cos\theta + \sin\theta\hat{\mathbf{a}}
\end{aligned}
$$

7 Interpolating unit quaternions

Suppose we have two orientations represented by $\mathbf{R}_0$ and $\mathbf{R}_1$ in $SO(3)$. Previous sections have shown that $\mathbf{R}_1 = \mathbf{R}_0\mathbf{R}_{(\theta,\mathbf{a})}$, and that a smooth intermediate path is obtained with $\mathbf{R}(t) = \mathbf{R}_0\mathbf{R}_{(t\theta,\mathbf{a})}$, for t in $[0, 1]$.
Let $\mathbf{q}_0 = \cos\phi + \sin\phi\hat{\mathbf{u}}$ and $\cos\theta + \sin\theta\hat{\mathbf{a}}$ correspond to the rotations $\mathbf{R}_0$ and $\mathbf{R}_{(\theta,\mathbf{a})}$.
Let $\mathbf{q}(t) = \mathbf{q}_0(\cos(t\theta) + \sin(t\theta)\hat{\mathbf{a}})$. Then $\mathbf{q}(0) = \mathbf{q}_0$ and $\mathbf{q}(1) = \mathbf{q_1}$ and

$$
\begin{aligned}
\mathbf{q}(t) &= \cos(t\theta)\mathbf{q}_0 + \sin(t\theta)\mathbf{q}_0\hat{\mathbf{a}} \\
&= \cos(t\theta)\mathbf{q}_0 - \frac{\cos\theta\sin(t\theta)}{\sin\theta}\mathbf{q}_0 + \frac{\cos\theta\sin(t\theta)}{\sin\theta}\mathbf{q}_0 + \sin(t\theta)\mathbf{q}_0\hat{\mathbf{a}} \\
&= \frac{1}{\sin\theta}((\sin\theta\cos(t\theta) - \cos\theta\sin(t\theta))\mathbf{q}_0 + \cos\theta\sin(t\theta)\mathbf{q}_0 + \sin\theta\sin(t\theta)\mathbf{q}_0\hat{\mathbf{a}}) \\
&= \frac{1}{\sin\theta}(\sin(\theta - t\theta)\mathbf{q}_0 + \sin(t\theta)\mathbf{q}_0(\cos\theta + \sin\theta\hat{\mathbf{a}})) \\
&= \frac{\sin(\theta - t\theta)}{\sin\theta}\mathbf{q}_0 + \frac{\sin(t\theta)}{\sin\theta}\mathbf{q}_1
\end{aligned}
$$

References

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